HOLOMORPHIC EXTENSION OF EIGENFUNCTIONS

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Abstract. Let $X = G/K$ be a Riemannian symmetric space of non-compact type. We prove a theorem of holomorphic extension for eigenfunctions of the Laplace-Beltrami operator on X, by techniques from the theory of partial differential equations.

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1. Introduction

Let X be a Riemannian symmetric space of non-compact type. Then $X = G/K$, where G is a connected semisimple Lie group and K a maximal compact subgroup. We choose the group G such that it is contained in a complexification $G_{\mathbb{C}}$, and we denote by $K_{\mathbb{C}} \subset G_{\mathbb{C}}$ the complexification of K. The symmetric space $X_{\mathbb{C}} = G_{\mathbb{C}}/K_{\mathbb{C}}$ carries a natural complex structure, and it contains X as a totally real submanifold.

We are interested in eigenfunctions of the Laplace-Beltrami operator Δ on X. Since this operator is elliptic and G-invariant, every eigenfunction admits a holomorphic extension to some open G-invariant neighborhood of X in $X_{\mathbb{C}}$. The G-orbits in $X_{\mathbb{C}}$ are generally difficult to parametrize, but let us recall that a particular G-invariant open neighborhood Ξ of X, for which the orbit structure is compellingly simple, has been proposed in [1]. It is commonly called the *complex crown* of X, and it has been thoroughly investigated in recent years. See for example [**?**, **?**, 3, 4, 10, 11, 12]. In the present paper we show that every eigenfunction for Δ extends holomorphically to Ξ.

Our result generalizes a result from [11] that every joint eigenfunction for the full set of invariant differential operators on X extends holomorphically to Ξ. The proof given in [11] invokes the Helgason conjecture, affirmed in [9] by micro-local analysis. Our proof is considerably simpler. The crucial step is an application of a theorem from the theory of analytic partial differential equations. This theorem asserts the existence of a holomorphic extension to solutions which are holomorphic on one side of a non-characteristic surface.

At the end of the paper a further generalization is given to functions on G, which are eigenfunctions for the Casimir operator and right-K-finite.

2. Notation

We denote by $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ the Lie algebra of G and its Cartan decomposition. We choose a maximal abelian subspace $\mathfrak{a}$ in $\mathfrak{p}$ and denote by $\Sigma \subset \mathfrak{a}^*$ the corresponding system of restricted roots. The root spaces in $\mathfrak{g}$ are denoted by $\mathfrak{g}^\alpha$, where $\alpha \in \Sigma$, and by $\Sigma^+ \subset \Sigma$ we denote a positive system. The centralizer of $\mathfrak{a}$ in K is $M = Z_K(\mathfrak{a})$, and the Weyl group is $W = N_K(\mathfrak{a})/M$, where $N_K(\mathfrak{a})$ is the normalizer.

Recall the definition of the complex crown Ξ of X. We set

$$\Omega := \{Y \in \mathfrak{a} \mid |\alpha(Y)| < \frac{\pi}{2}, \forall \alpha \in \Sigma\}.$$

Then

$$\Xi := G\exp(i\Omega)K_{\mathbb{C}} = \{g\exp(iY)\cdot x_0 \mid g \in G, Y \in \Omega\} \subset X_{\mathbb{C}}.$$

Here x_0 denotes the standard base point $eK_{\mathbb{C}}$ in $X_{\mathbb{C}}$.

3. Results

Lemma 3.1. *The G-invariant crown Ξ is an open subset of $X_{\mathbb{C}}$. The surjective map*

$$\Phi : G \times \Omega \to \Xi, \quad (g, Y) \mapsto g\exp(iY)\cdot x_0$$

is real analytic, and the topology of Ξ is identical to the quotient topology with respect to this map.

Let Ω^+ be the intersection of Ω with the positive open chamber in $\mathfrak{a}$, and let $\Xi' = \Phi(G \times \Omega^+)$. Then Ξ' is open and dense in Ξ, and

$$\Phi' : G/M \times \Omega^+ \to \Xi', \quad (gM, Y) \mapsto \Phi(g, Y)$$

is a diffeomorphism.

Proof. Apart from the statement about the topology of Ξ, this can be found in [11], §4. For the topological statement we need to prove that a subset of Ξ is open if its preimage is open. It suffices to prove the following. Let $z_n \to z \in \Xi$ be a converging sequence. Then there exists a subsequence of the form $z_j = \Phi(g_j, Y_j)$ with converging sequences $g_j \to g \in G$ and $Y_j \to Y \in \Omega$. It follows from [1], Propositions 1 and 7, that there exist sequences g_n in G, k_n in K and Y_n in Ω such that $z_n = \Phi(g_n k_n, Y_n)$, and such that g_n and $k_n\exp(iY_n)\cdot x_0$ both converge. By passing to a subsequence, we may assume that k_j converges, and since $Y \mapsto \exp(iY)\cdot x_0$ is a diffeomorphism of Ω onto its image, it then follows that Y_j converges in Ω. ∎

Theorem 3.2. *Let $f \in C^\infty(X)$ be an eigenfunction for Δ. Then f extends to a holomorphic function on Ξ.*

Proof. As Δ is elliptic, the regularity theorem for elliptic differential operators (see [6], Theorem 7.5.1) implies that f is an analytic function. As such it has an extension to a holomorphic function on some open neighborhood U_0 of x_0 in Ξ. It follows from the proof in [6], that U_0 can be chosen independently of f, that is, *every* eigenfunction can be holomorphically extended to U_0 (the radius of convergence obtained in the proof depends only on the corresponding radii for the coefficients of the differential operator). In particular, it follows from the fact that Δ is G-invariant, that $L_g f$ extends to U_0 for all $g \in G$. The union U of the

G-translated sets $L_{g^{-1}}(U_0)$ is then a G-invariant open neighborhood of X in Ξ, to which f extends.

We now consider the open dense subset $\Xi' \subset \Xi$ from Lemma 3.1. The intersection $U \cap \Xi'$ is non-empty, open and G-invariant. Let $Y_0 \in \Omega^+$, and for $r > 0$ let B_r denote the open ball in $\mathfrak{a}$ of radius r, centered at Y_0. If $B_r \subset \Omega^+$, then we define an open set

$$T_r = G \exp(iB_r) \cdot x_0 \subset \Xi',$$

which we regard as a G-invariant 'circular tube' in Ξ'. We claim that if f extends holomorphically to a set containing some circular tube $T_r \subset \Xi'$ centered at Y_0, then it extends to all circular tubes in Ξ' centered at Y_0. Since Y_0 was arbitrary, and since Ω^+ is simply connected, it follows from this claim that f extends holomorphically from $U \cap \Xi'$ to Ξ'.

In order to establish the claim we use Theorem 9.4.7 of [8], due to Zerner [13]. We write $\Delta_{\mathbb{C}}$ for the extension of Δ to a $G_{\mathbb{C}}$-invariant holomorphic differential operator on $X_{\mathbb{C}}$. Obviously, the holomorphic extension that we seek will be an eigenfunction for $\Delta_{\mathbb{C}}$ on Ξ. It follows from Lemma 3.1 that each circular tube T_r, for which the closure is contained in Ξ', has real-analytic boundary ∂T_r. In order to apply Zerner's theorem it suffices to establish that ∂T_r is non-characteristic for $\Delta_{\mathbb{C}}$, for all such tubes. By G-invariance, it suffices to consider boundary points $x \in \partial T_r$ with $x \in \exp(i\Omega) \cdot x_0$. Recall from [11], p. 207, that when $x \in \exp(i\Omega) \cdot x_0$ we have a complex-linear isomorphism

$$\mathfrak{p}_{\mathbb{C}} \ni Z \mapsto \tilde{Z}_x \in T_x\Xi, \tag{3.1}$$

where $\tilde{Z}$ is the holomorphic vector field on $X_{\mathbb{C}}$ given by

$$\tilde{Z}_x \varphi = L_Z \varphi(x) = \frac{d}{dz} \varphi(\exp(-zZ)x)|_{z=0}.$$

In this isomorphism the tangent space at x of the boundary ∂T_r will then be a real hyperplane given by an equation $\operatorname{Re} \zeta(Z) = 0$ for some cotangent vector $\zeta \in \mathfrak{p}_{\mathbb{C}}^*$. Since the tube T_r is G-invariant, it follows that $\operatorname{Re} \zeta$ annihilates Z for all $Z \in \mathfrak{p}$, so ζ is purely imaginary on $\mathfrak{p}$.

Let $(X_j^\alpha)_{\alpha \in \Sigma^+, 1 \le j \le m_\alpha}$ together with $Y_1, \dots, Y_r \in \mathfrak{a}$ be an orthonormal basis for $\mathfrak{p}$ such that $X_j^\alpha \in \mathfrak{p}_\alpha := [\mathfrak{g}^\alpha + \mathfrak{g}^{-\alpha}] \cap \mathfrak{p}$. Here $m_\alpha = \dim \mathfrak{p}_\alpha$ as usual. In the universal enveloping algebra we have

$$\Delta = \sum_{\alpha \in \Sigma^+} \sum_{j=1}^{m_\alpha} (X_j^\alpha)^2 + \sum_{i=1}^{r} Y_i^2$$

with respect to the right action, where functions on G/K are regarded as a right K-invariant function on G. Observe that modulo $\mathfrak{k}$,

$$\operatorname{Ad}(a)^{-1}(X_j^\alpha) = \cosh(\alpha(\log a)) X_j^\alpha$$

for $a \in A$ (see [11] p. 207), and hence

$$R_{X_j^\alpha} f(a) = -[\cosh \alpha(\log a)]^{-1} L_{X_j^\alpha} f(a).$$

It follows that

$$\Delta = \sum_{\alpha \in \Sigma^+} \sum_{j=1}^{m_\alpha} [\cosh \alpha(\log a)]^{-2} (X_j^\alpha)^2 + \sum_{i=1}^{r} Y_i^2 \tag{3.2}$$

at $z = a \cdot x_0 \in X$, with respect to the left action.

By analytic continuation, the same equation holds as well for $\Delta_{\mathbb{C}}$ and $a \in A_{\mathbb{C}}$. In particular, at $x = \exp(iY) \cdot x_0$ we obtain

$$\Delta_{\mathbb{C}} = \sum_{\alpha \in \Sigma^+} \sum_{j=1}^{m_\alpha} [\cos \alpha(Y)]^{-2} [(\tilde{X}_j^\alpha)_x]^2 + \sum_{i=1}^{r} [(\tilde{Y}_i)_x]^2.$$

Note that the condition that x belongs to the crown precisely ensures that $\cos \alpha(Y) \neq 0$, so that the expression makes sense. As ζ is purely imaginary on $\mathfrak{p}$, it follows that all terms in the above sum are ≤ 0 when applied to ζ. Thus the principal symbol of $\Delta_{\mathbb{C}}$ is non-zero at ζ, and the boundary of T_r is non-characteristic. It follows that Zerner's theorem can be applied, so that f extends holomorphically to Ξ'.

For the extension to the full set Ξ we shall apply Bochner's theorem (see [7], Theorem 2.5.10). ¿From what we have seen so far, for all $g \in G$ the function

$$f_g : \mathfrak{a} \to \mathbb{C}, \quad Y \mapsto f(g \exp(Y) \cdot x_0)$$

extends to a holomorphic function on a tubular neighborhood $\mathfrak{a} + i\omega$ of $\mathfrak{a}$ in $\mathfrak{a}_{\mathbb{C}}$, and also to $\mathfrak{a} + i\Omega^+$. For elements $w \in N_K(\mathfrak{a})$ we have

$$f_g(\mathrm{Ad}(w)Y) = f_{gw}(Y).$$

It follows that f_g extends to a holomorphic function on each Weyl conjugate of $\mathfrak{a} + i\Omega^+$, hence to $\mathfrak{a} + i\Omega'$, where $\Omega' = \cup \mathrm{Ad}(w)(\Omega^+)$ is the set of regular elements in Ω. Now Bochner's theorem implies that f_g extends to a holomorphic function on the tube over the convex hull of $\omega \cup \Omega'$, that is, to $\mathfrak{a} + i\Omega$. Furthermore, $g \mapsto f_g$ is continuous into $H(\mathfrak{a} + i\Omega)$ (with standard topology), since it is continuous into the space $H(\mathfrak{a} + i(\omega \cup \Omega'))$ which by Bochner's theorem is topologically isomorphic.

Recall that for all $g, g' \in G$ and $Y, Y' \in \Omega$ we have

$$g \exp(iY) \cdot x_0 = g' \exp(iY') \cdot x_0$$

if and only if there exists $w \in N_K(\mathfrak{a})$ and $k \in Z_K(Y)$ with $g' = gkw$ and $Y' = \mathrm{Ad}(w^{-1})Y$. It follows easily that by

$$g \exp(iY) \cdot x_0 \mapsto f_g(Y)$$

we obtain a well-defined extension of f on Ξ. The topological statement in the first part of Lemma 3.1 implies that this extension is continuous. Since the extension is holomorphic on Ξ', it must be holomorphic everywhere. $\square$

We list some easy consequences of the preceding theorem and its proof. ¿From the Iwasawa decomposition $G = NAK$ associated to the positive system Σ^+, we obtain the familiar horospherical projection $x \mapsto H(x) \in \mathfrak{a}$, defined by $x \in N \exp H(x) \cdot x_0$ for $x \in X$. For each $\lambda \in \alpha^*_{\mathbb{C}}$ the function

$$x \mapsto e^{\lambda(H(x))}$$

on X is an eigenfunction for Δ, hence extends to a holomorphic function on Ξ. We obtain:

Corollary 3.3. *The projection $H : X \to \mathfrak{a}$ extends to a holomorphic map $\Xi \to \mathfrak{a}_{\mathbb{C}}$. Moreover, $\Xi \subset N_{\mathbb{C}} A_{\mathbb{C}} K_{\mathbb{C}}$.*

Proof. Let $h_\lambda(z)$ denote the analytic continuation of $e^{\lambda(H(x))}$. Since $h_{-\lambda}(z) = h_\lambda(z)^{-1}$ we conclude that $h_\lambda(z) \neq 0$. As Ξ is simply connected, the analytic continuation of $\lambda(H(x))$ is obtained by taking logarithms, and the first statement follows. For the last statement, we note that once the Iwasawa A-component allows an analytic continuation, then so does the N-component. Indeed, knowing the A-component, we can determine the N-component of $x \in X$ from $\theta(x)x^{-1}$, where θ denotes the Cartan involution. $\square$

The preceding corollary was obtained for classical groups in [10]. The general case follows from results established in [12], [2] and [4] with [5].

Let $\omega \subset \Omega$ be open, convex and W-invariant, and let $T_\omega \subset \Xi$ denote the open set

$$T_\omega = G \exp(i\omega) \cdot x_0.$$

Corollary 3.4. *Let $f \in C^\infty(X)$ and let P be a non-trivial polynomial of one variable. If $P(\Delta)f$ extends to a holomorphic function on T_ω, then so does f.*

Proof. By treating the factors of P successively we may assume that $P(\Delta) = \Delta - \lambda$. The proof of Theorem 3.2 can then be repeated. $\square$

The following generalization is more far-reaching. We denote by $\mathrm{C} \in \mathcal{Z}(\mathfrak{g})$ the Casimir element of $\mathfrak{g}$.

Theorem 3.5. *Let $f \in C^\infty(G)$ be a right K-finite eigenfunction of* C. *Then f extends to a holomorphic function on*

$$\tilde{\Xi} := G \exp(i\hat{\Omega}) K_{\mathbb{C}} \subset G_{\mathbb{C}}.$$

Proof. Recall that f being K-finite means that the translates $R_k f$ by $k \in K$ span a finite dimensional space, which is then a representation space for K. We may assume that it is irreducible, and then f is an eigenfunction for the Casimir element $\mathrm{C}_{\mathfrak{k}}$ of $\mathfrak{k}$, acting from the right. The operator $\mathrm{C}+2\mathrm{C}_{\mathfrak{k}}$ is elliptic, so it follows that f is real analytic. The proof of Theorem 3.2 can now be repeated, with the following changes.

In Lemma 3.1 we replace the map Φ by

$$\tilde{\Phi} : G \times \Omega \times K_{\mathbb{C}} \to \tilde{\Xi}, \quad (g, Y, k) \mapsto g \exp(iY)k,$$

and we define $\tilde{\Phi}'$ as before, but now on $(G \times \Omega^+ \times K_{\mathbb{C}})/M$, where M acts on the first and last factor, from the right and left, respectively.

The G-invariant tubes $T_r \subset \Xi$ are replaced by their $G \times K_{\mathbb{C}}$-invariant preimages $\tilde{T}_r = T_r K_{\mathbb{C}} \subset \tilde{\Xi}$. The map $Z \mapsto L_Z$ from $\mathfrak{p}_{\mathbb{C}}$ onto $T_x\Xi$ in (3.1) is replaced by $Z \oplus U \mapsto L_Z + R_U$ from $\mathfrak{g}_{\mathbb{C}} = \mathfrak{p}_{\mathbb{C}} \oplus \mathfrak{k}_{\mathbb{C}}$ onto $T_x\tilde{\Xi}$. Here $x \in \exp(i\Omega)$. The cotangent vector ζ, normal to $\tilde{T}_r$ at x is zero on $\mathfrak{k}_{\mathbb{C}}$ and purely imaginary on $\mathfrak{p}$, by the same argument as before.

Since f is a $\mathrm{C}_{\mathfrak{k}}$-eigenfunction, the action of C on it differs only by a constant from that of the operator Δ described in (3.2). Hence $\partial\tilde{T}_r$ is non-characteristic for C, and the application of Zerner's theorem goes through. The rest of the argument is essentially unchanged. $\square$

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